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The effect of mainflow transverse velocities in linear stability theory

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In studying the stability of the boundary layer with surface mass injection, a generalized version of the Orr–Sommerfeld equation was derived which takes account of the transverse velocity component in the mainflow. The new terms in the generalized Orr–Sommerfeld equation are inversely proportional to the Reynolds number. The resulting eigenvalue problem was solved numerically for a wide range of values of the mass injection intensity. It was found that the critical Reynolds number (based on the distance from the leading edge) decreases with increasing mass injection. The deviations between the critical Reynolds numbers from the generalized and conventional Orr–Sommerfeld equations have a different sign at low injection intensities from that at high injection intensities.

Introduction

The conventional formulation of the linear theory of hydrodynamic stability, leading to the Orr–Sommerfeld equation, is based on a model which sets aside the transverse velocity component in the mainflow. It is assumed that the mainflow can be regarded as a parallel flow consisting solely of the streamwise velocity component, with the other velocity components being zero. Such a model is fulfilled exactly in fully developed duct flows, whereas for boundary layers it is an approximation.

The characterization of a boundary-layer flow as a uni-component flow may be intuitively acceptable when there is no mass transfer at the bounding surface. However, in the presence of surface mass transfer, the magnitude of the transverse velocity may be substantially augmented, so that a more detailed analysis of the contribution of the transverse velocity to the stability problem is warranted.

In the present paper, a generalized version of the Orr–Sommerfeld equation is derived which takes account of the transverse velocity component in the mainflow. The derivation lifts the assumption of a strictly parallel mainflow and utilizes only the magnitude ordering of boundary-layer theory. The resulting governing equation for the disturbance amplitude resembles the Orr–Sommerfeld equation, but contains additional terms involving the mainflow transverse velocity and its second derivative, all such terms being inversely proportional to the Reynolds number. The influence of the new terms should therefore be accentuated in cases in which the critical Reynolds number is relatively small. For surface mass transfer, it is known that injection decreases the critical Reynolds number whereas suction has the opposite effect.

Solutions of the aforementioned generalized Orr–Sommerfeld equation were carried out for the boundary layer on a flat plate. In view of the foregoing, consideration was focused on the case of mass injection at the plate surface. A wide range of injection velocities was investigated. The solutions were carried out using an extension of the finite-difference method of Thomas (1953), the extension being necessitated by the presence of the third derivative of the disturbance amplitude in the generalized Orr–Sommerfeld equation. Numerical results are presented for the critical Reynolds numbers and for the neutral stability curves. This information is compared with corresponding results from the conventional Orr–Sommerfeld equation, where the influence of the transverse velocity is neglected.

The generalized Orr–Sommerfeld equation, which takes account of the mainflow transverse velocity, reduces to a somewhat simpler form for the asymptotic suction boundary layer. In this relatively simple case, the mainflow velocity field is independent of the streamwise co-ordinate and the transverse velocity is constant everywhere. The reduced form of the generalized Orr–Sommerfeld equation is stated by Hughes & Reid (1965) in their definitive treatment of the asymptotic suction boundary layer and by Alekseev & Korotkin (1966). Since the asymptotic suction boundary layer is characterized by a very high critical Reynolds number, the term involving the transverse velocity plays a negligible role.

The generalized Orr–Sommerfeld equation

The starting point of the analysis is the Navier–Stokes equations for incompressible two-dimensional time-dependent fluid motion. Consider a boundary-layer flow with velocity components $\hat{u}$ and $\hat{v}$ in the streamwise and transverse directions (x and y, respectively) and with static pressure distribution $\hat{p}$. If u, v, p denote mainflow quantities and u', v', p' are the corresponding disturbances, then

$$\hat{u} = u+u', \quad \hat{v} = v+v', \quad \hat{p} = p+p'. \tag{1}$$

Substitution of (1) into the Navier–Stokes equations, followed by subtraction of the mainflow and neglect of squares of disturbance quantities, leads to

$$\frac{\partial u'}{\partial t}+u\frac{\partial u'}{\partial x}+u'\frac{\partial u}{\partial x}+v\frac{\partial u'}{\partial y}+v'\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p'}{\partial x}+\nu\left(\frac{\partial^2 u'}{\partial x^2}+\frac{\partial^2 u'}{\partial y^2}\right), \tag{2}$$

$$\frac{\partial v'}{\partial t}+u\frac{\partial v'}{\partial x}+u'\frac{\partial v}{\partial x}+v\frac{\partial v'}{\partial y}+v'\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p'}{\partial y}+\nu\left(\frac{\partial^2 v'}{\partial x^2}+\frac{\partial^2 v'}{\partial y^2}\right). \tag{3}$$

Equations (2) and (3) retain all of the terms involving $\partial u/\partial x$, v, $\partial v/\partial x$, and $\partial v/\partial y$ that are normally deleted in the derivation of the conventional Orr–Sommerfeld equation.

The pressure p' may now be eliminated from (2) and (3) by cross differentiation and subtraction. The resulting equation is then simplified by noting that $\partial u/\partial x + \partial v/\partial y = 0$ and that $\partial^2 u/\partial x^2 \ll \partial^2 u/\partial y^2$, $\partial^2 v/\partial x^2 \ll \partial^2 v/\partial y^2$. This magnitude ordering of the second derivatives is precisely that which is used in the analysis of the mainflow, so that no new approximations are being introduced here. According to boundary-layer theory, the second derivatives that are deleted are of order δ^2 (δ = boundary-layer thickness) compared with those that are retained. The equation resulting from the operations just described is

$$\frac{\partial^2 u'}{\partial y\,\partial t} - \frac{\partial^2 v'}{\partial x\,\partial t} + u\left(\frac{\partial^2 u'}{\partial y\,\partial x} - \frac{\partial^2 v'}{\partial x^2}\right) + v\left(\frac{\partial^2 u'}{\partial y^2} - \frac{\partial^2 v'}{\partial x\,\partial y}\right) + v'\frac{\partial^2 u}{\partial y^2} - u'\frac{\partial^2 v}{\partial y^2}$$
$$= \nu\left\{\frac{\partial}{\partial y}\left(\frac{\partial^2 u'}{\partial x^2} + \frac{\partial^2 u'}{\partial y^2}\right) - \frac{\partial}{\partial x}\left(\frac{\partial^2 v'}{\partial x^2} + \frac{\partial^2 v'}{\partial y^2}\right)\right\}. \tag{4}$$

Compared with the corresponding stage in the derivation of the conventional Orr–Sommerfeld equation, three additional terms involving v and $\partial^2 v/\partial y^2$ may be identified in (4).

The next step is to evaluate the perturbation velocities via the continuity relations $u' = \partial\psi'/\partial y$ and $v' = -\partial\psi'/\partial x$. ψ' is the stream function of the disturbance, which is assumed to have the form of a plane wave of amplitude ϕ travelling in the streamwise direction

$$\psi'(x,y,t) = \phi(y)\exp\left[i\alpha'(x - c't)\right]. \tag{5}$$

Upon substituting u' and v' from (5) into (4), there emerges a differential equation for the disturbance amplitude ϕ which is fourth order in y, with all derivatives in x dropping out.

To proceed, it is convenient to scale y by the local boundary-layer thickness δ corresponding to $u/u_\infty = 1-\epsilon$ (u_∞ is the free stream velocity; ϵ is a small positive number, taken as 0·001 in the present study). The scaled y co-ordinate and other relevant dimensionless quantities are introduced according to the definitions

$$Y = y/\delta, \quad D^n = d^n/dY^n, \tag{6a}$$

$$U = u/u_\infty, \quad V = v/u_\infty, \quad c = c'/u_\infty, \quad \alpha = \alpha'\delta, \quad R = u_\infty\delta/\nu, \tag{6b}$$

with the result that

$$(U-c)(D^2-\alpha^2)\phi - \left(\frac{\partial^2 U}{\partial Y^2}\right)\phi - \frac{i}{\alpha}\left[V(D^3-\alpha^2 D)\phi - \left(\frac{\partial^2 V}{\partial Y^2}\right)D\phi\right]$$
$$= -\frac{i}{\alpha R}(D^4 - 2\alpha^2 D^2 + \alpha^4)\phi. \tag{7}$$

Equation (7) differs from the conventional Orr–Sommerfeld equation by the presence of the rightmost group of terms on the left-hand side. As pointed out by Reid (1965), the retention of the transverse velocity terms requires, for consistency, that the terms $(-2\alpha^2 D^2 + \alpha^4)\phi$ also be retained. The latter terms are sometimes deleted in the asymptotic-type analysis of the conventional Orr–Sommerfeld equation.

The terms in (7) involving U, V and their derivatives will now be evaluated from the mainflow solution. From boundary-layer theory, it is well established that for laminar flow over a flat plate in the presence of surface mass transfer, similarity solutions are possible only when the injection or suction velocity v_w varies with the axial co-ordinate x as

$$v_w/u_\infty = -\tfrac{1}{2}(\nu/u_\infty x)^{\frac{1}{2}}F(0), \tag{8}$$

where $F(0)$ is a prescribed constant whose magnitude specifies the intensity of the mass transfer. Positive and negative values of $F(0)$ correspond, respectively, to suction and injection, while $F(0) = 0$ for an impermeable boundary. For the similarity solution, the utilization of a similarity variable η and a reduced stream function $F(\eta)$, respectively defined as

$$\eta = y(u_\infty/\nu x)^{\frac{1}{2}}, \quad F(\eta) = \psi(\nu x u_\infty)^{-\frac{1}{2}}, \tag{9}$$

leads to the Blasius equation

$$\frac{d^3F}{d\eta^3} + \tfrac{1}{2}F\frac{d^2F}{d\eta^2} = 0 \tag{10}$$

with boundary conditions F = constant and $dF/d\eta = 0$ at $\eta = 0$, and $dF/d\eta = 1$ as $\eta \to \infty$.

The solution for F and its derivatives depends parametrically on $F(0)$ and is readily found by numerical techniques. From such solutions, all the information needed for the stability analysis can be extracted. The boundary-layer thickness δ, defined as the distance from the plate surface where $u/u_\infty = 0{\cdot}9990$, is given by

$$\delta = \eta_\delta(\nu x/u_\infty)^{\frac{1}{2}}, \tag{11}$$

where η_δ corresponds to the η value at which $dF/d\eta = 0{\cdot}9990$. Furthermore, the mainflow velocities and velocity derivatives that appear in the generalized Orr–Sommerfeld equation (7) can be represented as

$$U = \frac{dF}{d\eta}, \quad \frac{\partial^2 U}{\partial Y^2} = \Gamma(\eta), \quad V = \frac{\Omega(\eta)}{R}, \quad \frac{\partial^2 V}{\partial Y^2} = \frac{\chi(\eta)}{R}, \tag{12a}$$

in which

$$\Gamma = -\tfrac{1}{2}\eta_\delta^2 F\frac{d^2F}{d\eta^2}, \quad \Omega = -\tfrac{1}{2}\eta_\delta\left(F - \eta\frac{dF}{d\eta}\right), \quad \chi = \tfrac{1}{2}\eta_\delta^3(1 - \tfrac{1}{2}\eta F)\frac{d^2F}{d\eta^2} \tag{12b}$$

and $Y = \eta/\eta_\delta$.

In view of the two right-hand members of (12*a*), it is seen that the new terms in the generalized Orr–Sommerfeld equation (7) are proportional to $1/R$. Consequently, the influence of the transverse velocity should be most strongly felt in flows in which the critical Reynolds number is relatively low.

The disturbance amplitude ϕ is governed by (7), supplemented by (12*a*) and (12*b*). The disturbance velocities u' and v' must vanish at the plate surface and in the free stream outside the boundary layer. In terms of ϕ these boundary conditions can be expressed as

$$\phi = D\phi = 0 \quad \text{at} \quad Y = 0, \tag{13}$$

$$\phi \sim \exp(-\alpha Y) \quad \text{for} \quad Y \geqslant 1, \tag{14}$$

where (14) is a solution of the inviscid form of (7).

The complex wave velocity c appearing in (7) is made up of components c_r and c_i. The flow is stable, neutrally stable, or unstable relative to the disturbance depending on whether c_i is less than, equal to, or greater than zero. The mathematical system consisting of (7), (13) and (14) is an eigenvalue problem which gives the relationship among α, c and R.

Solution of the eigenvalue problem

In approaching the solution of the eigenvalue problem, it is convenient to rephrase (7), evaluated using (12*a*) and (12*b*), in the form

$$D^4\phi + A_1 D^3\phi + A_2 D^2\phi + A_3 D\phi + A_4\phi = 0, \tag{15}$$

where

$$A_1 = -\Omega, \quad A_2 = \frac{\alpha R}{i}\left(\frac{dF}{d\eta} - c\right) - 2\alpha^2,$$

$$A_3 = \alpha^2\Omega + \chi, \quad A_4 = \alpha^4 - \frac{\alpha R}{i}\left[\alpha^2\left(\frac{dF}{d\eta} - c\right) + \Gamma\right]. \tag{16}$$

A finite-difference technique was employed in solving the eigenvalue problem defined by (15), (14) and (13).

For finite-difference solutions of the conventional Orr–Sommerfeld equation, it has been found advantageous to formulate the difference equations by employing the transformation of Thomas (1953). The particular usefulness of this transformation lies in the small truncation errors that result from the discretization. The truncation error is relatively large only for the third derivative, $D^3\phi$, and in the past this has not been a drawback since the conventional Orr–Sommerfeld equation does not contain the third derivative. On the other hand, the third derivative does appear in the generalized Orr–Sommerfeld equation and steps must be taken to remove it before going ahead with the difference formulation.

To proceed, a transformation to a new dependent variable θ is made following a suggestion of Fu (1967).

$$\phi = \theta \exp\left[-\frac{1}{4}\int_0^Y A_1\, dY\right]. \tag{17}$$

After a lengthy derivation, (15) transforms into

$$D^4\theta + B_1 D^2\theta + B_2 D\theta + B_3\theta = 0, \tag{18}$$

in which

$$\left.\begin{aligned} B_1 &= A_2 - \tfrac{3}{8}A_1^2 - \tfrac{3}{2}A_1', \\ B_2 &= A_3 - \tfrac{1}{2}A_1A_2 + \tfrac{1}{8}A_1^3 - A_1'', \\ B_3 &= A_4 - \tfrac{1}{4}A_1A_3 + \tfrac{1}{16}A_1^2A_2 - \tfrac{3}{256}A_1^4 + \tfrac{3}{32}A_1^2A_1' \\ &\qquad + \tfrac{3}{16}(A_1')^2 - \tfrac{1}{4}A_1''' - \tfrac{1}{4}A_1'A_2, \end{aligned}\right\} \tag{19}$$

where

$$\left.\begin{aligned} A_1' &= -\tfrac{1}{2}\eta_\delta^2\eta\, d^2F/d\eta^2, \\ A_1'' &= -\tfrac{1}{2}\eta_\delta^3(1 - \tfrac{1}{2}\eta F)\, d^2F/d\eta^2, \\ A_1''' &= \tfrac{1}{4}\eta_\delta^4(2F - \tfrac{1}{2}\eta F^2 + \eta\, dF/d\eta)\, d^2F/d\eta^2. \end{aligned}\right\} \tag{20}$$

The boundary conditions (13) and (14), when recast in terms of θ, become

$$\theta = D\theta = 0 \quad \text{at} \quad Y = 0, \tag{21}$$

$$\theta \sim \exp[-(\alpha - \tfrac{1}{4}A_1)\,Y] \quad \text{for} \quad Y \geqslant 1. \tag{22}$$

Since (18) does not contain the third derivative, Thomas's transformation can be employed to formulate the difference equations. The region $0 \leqslant Y \leqslant 1$ was subdivided into N equal intervals and the difference equations applied at $(N+1)$ discrete points, with special equations for the boundary points $Y = 0$ and $Y = 1$. This gives rise to a set of $(N+1)$ linear homogeneous complex algebraic equations. The requirement for a solution to exist is that the determinant $\mathscr{D}$ of the coefficient matrix be zero, which leads to a secular equation

$$f(\alpha, c, R) = 0. \tag{23}$$

For given α and R, the value of c satisfying (23) was determined by an iteration procedure in conjunction with the root finder technique of Muller (1956).

The objective of the calculations was to map out neutral stability curves and determine critical Reynolds numbers for a range of surface mass injection intensities $F(0)$ extending from zero (impermeable surface) to strong injection. The first case to be investigated was the impermeable surface ($F(0) = 0$). In this case, the influence of the transverse velocity is small and the initial values for the iteration procedure were taken from available solutions of the conventional Orr–Sommerfeld equation. Once the neutral curve and critical Reynolds number for $F(0) = 0$ were determined, then calculations for $F(0) = -0{\cdot}2$ were initiated, with the first input taken from the $F(0) = 0$ results; and so forth to $F(0) = -1{\cdot}0$.

A step size ΔY of $0{\cdot}01$ for the finite-difference formulation was found to be adequate for all cases. The calculations were performed with double precision arithmetic on an IBM 360/50 digital computer.

Results and discussion

The neutral stability curves were mapped out in terms of a Reynolds number R_1 and a wave-number α_1, which are based on the displacement thickness δ_1. These quantities are defined as

$$R_1 = u_\infty \delta_1/\nu, \quad \alpha_1 = \alpha' \delta_1, \tag{24}$$

$$\delta_1 = \int_0^\infty \left(1 - \frac{u}{u_\infty}\right) dy \quad \text{or} \quad \delta_1 (u_\infty/\nu x)^{\frac{1}{2}} = \int_0^\infty \left(1 - \frac{dF}{d\eta}\right) d\eta. \tag{25}$$

Representative neutral stability curves, corresponding to $F(0) = 0$, $-0{\cdot}4$, $-0{\cdot}8$, and $-1{\cdot}0$ are presented in figure 1. Neutral stability curves were determined for others $F(0)$ values, but are not included in order to preserve clarity.

Inspection of the figure indicates that as the injection intensity $|F(0)|$ increases from zero, there is a marked leftward shift of the curves toward lower values of R_1, suggesting a destabilization of the mainflow. For larger injection intensities, however, the trend toward lesser stability appears to reverse.

Further consideration of this apparent reversal will be deferred until later when the critical Reynolds numbers are presented.

Each neutral stability curve partitions the R_1, α_1 plane into two regions. In the region enclosed within the curve, $c_i > 0$ and disturbances are amplified, while outside the curve $c_i < 0$ and disturbances are damped. The figure shows that the height of the region of destabilization is enlarged as the injection intensity increases. It is also interesting to observe that for mass injection, i.e. $F(0) < 0$,

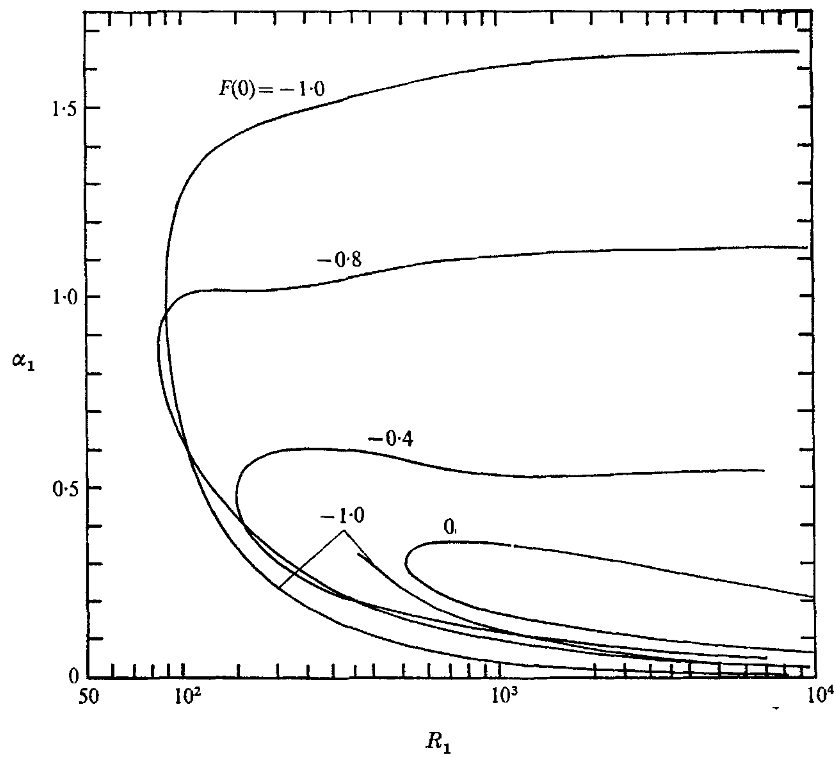

FIGURE 1. Representative neutral stability curves.

the upper branches of the neutral stability curves appear to tend to non-zero asymptotic values for large R_1. Therefore, even for $R_1 \to \infty$, there exists a finite range of wave-numbers in which disturbances are amplified. A similar behaviour occurs for boundary layers in adverse pressure gradients. In contrast, both branches of the neutral stability curve for the impermeable plate are asymptotic to zero as R_1 approaches infinity.

A rather unexpected finding was encountered during the course of the computations for $F(0) = -1$, which represents the largest injection intensity investigated. As shown in the figure, there are two lower branches of the neutral stability curve. One of these lower branches is a continuation of the upper branch; the other lower branch disappears at values of R_1 below 350. Along the former, c_r increases with increasing R_1, whereas there is an opposite variation along the latter. The significance of these dual lower branches remains uncertain.

Critical Reynolds numbers were determined from careful examination of neutral stability curves for eight values of the injection intensity $F(0)$ between 0 and -1. These critical values are denoted by $(R_1)_c$. In addition, critical values of the Reynolds number R_x based on the streamwise co-ordinate x were also evaluated by means of the relation

$$(R_x)^{\frac{1}{2}}_c = (R_1)_c/\delta_1(u_\infty/\nu x)^{\frac{1}{2}}, \tag{26}$$

where

$$R_x = u_\infty x/\nu. \tag{27}$$

The numerical results for $(R_1)_c$ and $(R_x)_c$ are listed in table 1 and are plotted in figure 2.

$F(0)$	$(R_1)_c$	$(R_x)^{\frac{1}{2}}_c$	$(\alpha_1)_c$	$(c_r)_c$	Y_c	$\delta_1(u_\infty/\nu x)^{\frac{1}{2}}$	$\delta_2(u_\infty/\nu x)^{\frac{1}{2}}$	H
0	510	296·4	0·305	0·402	0·204	1·7208	0·6641	2·5911
−0·2	259	132·2	0·383	0·457	0·257	1·9587	0·7233	2·7080
−0·4	149	65·7	0·472	0·505	0·315	2·2674	0·7912	2·8658
−0·6	98	36·5	0·630	0·550	0·379	2·6881	0·8697	3·0907
−0·7	89	30·0	0·737	0·566	0·411	2·9647	0·9138	3·2443
−0·8	85	25·7	0·855	0·578	0·444	3·3099	0·9616	3·4419
−0·9	86	22·9	0·962	0·589	0·480	3·7602	1·0138	3·7090
−1·0	89	20·3	0·990	0·619	0·531	4·3907	1·0710	4·0995

TABLE 1. Stability characteristics and related information

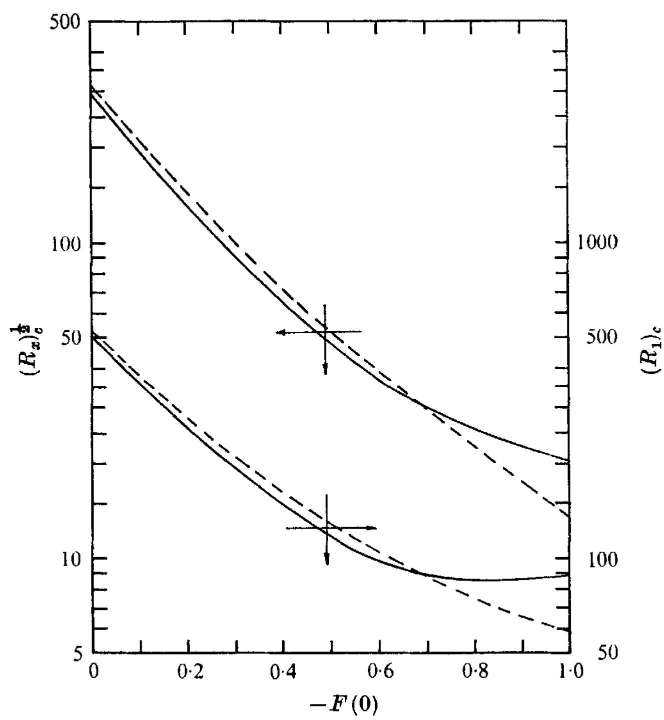

FIGURE 2. Variation of the critical Reynolds number with mass injection intensity. ——, generalized Orr–Sommerfeld equation; – – –, conventional Orr–Sommerfeld equation.

In the figure, the solid curves correspond to the solutions of the generalized Orr–Sommerfeld equation obtained during this investigation, whereas the dashed lines represent results from the conventional Orr–Sommerfeld equation (Tsou & Sparrow 1970). The results for $(R_x)_c$ are referred to the left-hand ordinate and those for $(R_1)_c$ to the right-hand ordinate.

Of the two representations of the stability limit, that is, $(R_x)_c$ and $(R_1)_c$, the former is believed to be the more physically relevant. From figure 2 it is seen that $(R_x)_c$ decreases monotonically as the injection intensity increases, for both

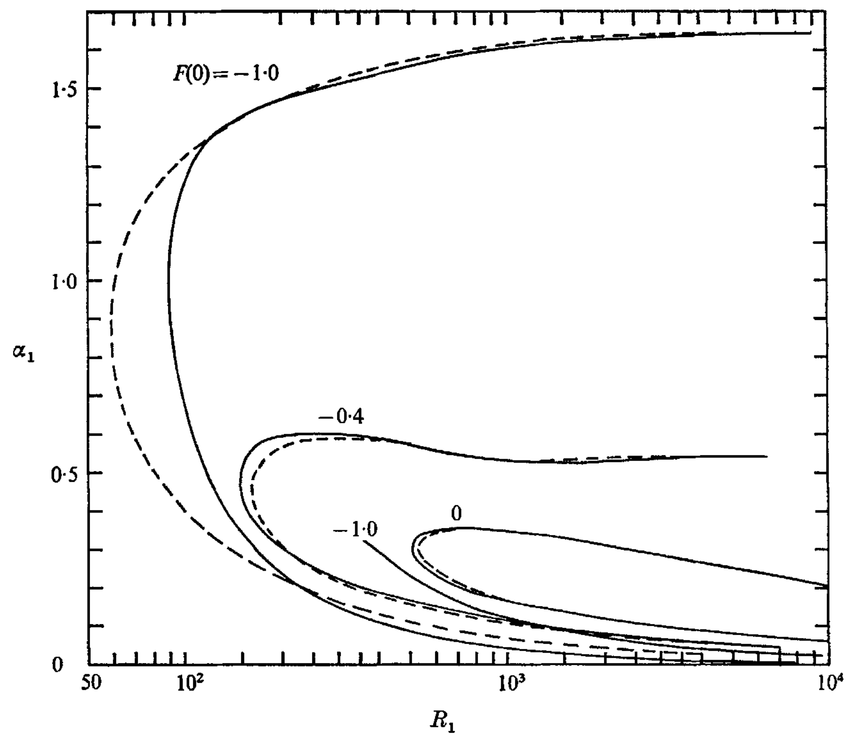

FIGURE 3. Comparison of neutral stability curves. ——, generalized Orr–Sommerfeld equation; – – –, conventional Orr–Sommerfeld equation.

the solutions of the generalized and conventional Orr–Sommerfeld equations. At small and intermediate injection intensities, the $(R_x)_c$ from the generalized Orr–Sommerfeld equation lie below those given by the conventional Orr–Sommerfeld equation, whereas the opposite behaviour is in evidence at the larger injection intensities. Thus, the role of the newly accounted terms involving the mainflow transverse velocity is different depending on the strength of the surface mass transfer.

With respect to the behaviour of the critical Reynolds number $(R_1)_c$, the results from the generalized Orr–Sommerfeld equation are seen to decrease with increasing injection intensity, attain a minimum, and then to increase slightly at the larger injection intensities. Thus, the trends in $(R_x)_c$ and $(R_1)_c$ appear to be at variance at the larger injection intensities. As indicated earlier, $(R_x)_c$ is believed to be the more relevant stability parameter, and its monotonic decrease with increasing blowing intensity is accepted as indicative of the effect of surface mass injection. The $(R_1)_c$ results from the conventional Orr–Sommerfeld equation decrease monotonically with injection intensity.

In addition to the Reynolds numbers $(R_1)_c$ and $(R_x)_c$, table 1 lists the values of the wave-number α_1 and wave velocity c_r at the critical condition, and the location of the critical layer Y_c. Both the wave-number and wave velocity corresponding to the critical condition increase with increasing injection intensity. The critical layer location Y_c, defined as the distance y/δ from the wall at which the local mainflow velocity $U(=u/u_\infty)$ is equal to $(c_r)_c$, is also greater at higher injection intensities. The last three columns of table 1 contain information on the displacement thickness δ_1, momentum thickness δ_2, and the ratio $H = \delta_1/\delta_2$.

A representative comparison of neutral stability curves from the solutions of the generalized and conventional Orr–Sommerfeld equations is made in figure 3, the results for the latter being taken from Tsou & Sparrow (1970). For a given injection intensity $F(0)$, the greatest deviations between the solid and dashed curves occur in the nose region, that is, where the Reynolds numbers are lowest. In the nose region, for low and moderate injection intensities, the dashed curves lie to the right of the solid curves. A reversed positioning is in evidence at the higher injection intensities.†

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† *Note added in proof.* Interested readers may wish to look at a paper by Barry & Ross (1970) which appeared in print when the present paper was being refereed. These authors concerned themselves with the impermeable plate.